

Implications of purely classical gravity for inflationary tensor modes

Amjad Ashoorioon¹, P. S. Bhupal Dev², and Anupam Mazumdar^{1,3}

¹ *Consortium for Fundamental Physics, Physics Department,
Lancaster University, LA1 4YB, United Kingdom.*

² *Consortium for Fundamental Physics, School of Physics and Astronomy,
University of Manchester, Manchester, M13 9PL, United Kingdom.*

³ *Niels Bohr Institute, Copenhagen University, Blegdamsvej-17, Denmark.*

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We discuss the implications of purely classical, instead of quantum, theory of gravity for the gravitational wave spectrum generated during inflation. We show that a positive detection of primordial gravitational waves will no longer suffice to determine the scale of inflation in this case – even a high-scale model of inflation can bypass the observational constraints due to large uncertainties in the initial classical amplitude of the tensor modes.

Primordial inflation is one of the most successful paradigms for the early Universe cosmology (for a review, see e.g., [1]), which has many observational consequences [2]. One of the predictions for inflation is the generation of stochastic primordial gravitational waves along with the matter perturbations [3]. Typically, matter perturbations are created from the initial vacuum fluctuations which are stretched outside the Hubble patch during inflation [4] (for reviews, see [5, 6]).

As in any quantum field theory in a time-dependent background, the initial choice of vacuum is typically obtained by imposing the quantum commutation relationships for creation and annihilation operators which satisfy the Wronskian condition, while confirming that the initial quantum state is the *least* excited state analogous to the plane wave solution emanating from deep inside the Hubble patch [5, 6]. Similar quantum calculations exist for the gravitational waves generated during inflation in which case one directly quantizes the tensor perturbations of the metric. Since the observed temperature anisotropy in the cosmic microwave background (CMB) radiation is very small: $\delta T/T \sim 10^{-5}$ [2], the treatment of linearized perturbation is a very good approximation for both matter and gravity sectors.

However, the assumption that gravity should also be quantized along with the matter perturbations is not yet based upon any observed phenomena¹. In fact, the entire framework of matter perturbations created during inflation can be carried out without quantizing the met-

ric fluctuations. What it means is that a pure de-Sitter background without matter cannot seed the temperature anisotropy in the CMB radiation [5, 6].

Therefore, a natural question arises – what if we had to treat the primordial gravity waves purely at the classical level by assuming that the space-time is indeed classical and we just quantize the matter part in a given space-time background. The aim of this letter is to explore such a possibility and to show what are the differences in predictions one would expect if gravity were to be treated classically, and in particular what would be the amplitude of the primordial gravitational waves.

We recall here that in a scalar-driven inflationary model, the tensor modes are generated from tensor fluctuations of the metric [5, 6]:

$$ds_T^2 = a^2(\tau) (d\tau^2 - [\delta_{ij} + h_{ij}] dx^i dx^j) \quad (1)$$

with $|h_{ij}| \ll 1$, where h_{ij} is a symmetric three-tensor field satisfying $h_i^i = 0 = h_{ij}{}^{,j}$ whose dynamics could be determined by expanding the Einstein-Hilbert action to second order:

$$S_T^{(1)} = \frac{M_p^2}{64\pi} \int d\tau d^3\mathbf{x} \, a^2(\tau) \, \partial_\mu h^i{}_j \, \partial^\mu h^j{}_i, \quad (2)$$

where M_p is the usual Planck mass. It is possible to reformulate the tensor action given by Eq. (2) to give it the appearance of a Minkowski-space theory with variable mass term by introducing the re-scaled variable $P^i{}_j$:

$$P^i{}_j(x) = \sqrt{\frac{M_p^2}{32\pi}} a(\tau) h^i{}_j(x), \quad (3)$$

whose dynamics is governed by

$$S_T^{(2)} = \frac{1}{2} \int d\tau d^3\mathbf{x} \times \left(\partial_\tau P_i{}^j \partial^\tau P^i{}_j - \delta^{rs} \partial_r P_i{}^j \partial_s P^i{}_j + \frac{a''}{a} P_i{}^j P^i{}_j \right)$$

¹ Whether gravity is truly quantum or classical is still an open issue (see [7, 8] for some discussions). For a consistent treatment, we often argue that gravity must be quantized along with the matter sector. However, even if gravity is treated classically, there is a hint that it might be possible to address non-singular blackhole and cosmological solutions, found very recently in the context of higher order *infinite-derivative* theories of gravity [9].

and is different from $S_T^{(1)}$ by a total time derivative. One can decompose P^i_j into its Fourier components:

$$P^i_j = \sum_{\lambda=+, \times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} p_{\mathbf{k},\lambda}(\tau) \epsilon^i_j(\mathbf{k}; \lambda) e^{i\mathbf{k}\cdot\mathbf{y}}, \quad (4)$$

where the sum is over two independent polarization states, usually denoted as $\lambda = +, \times$. $\epsilon^i_j(\mathbf{k}; \lambda)$ is the polarization tensor satisfying the following conditions:

$$\begin{aligned} \epsilon_{ij} &= \epsilon_{ji}, \quad \epsilon^i_i = 0, \quad k^i \epsilon_{ij} = 0, \quad \text{and} \\ \epsilon^i_j(\mathbf{k}; \lambda) \epsilon^{j*}_i(\mathbf{k}; \lambda') &= \delta_{\lambda\lambda'}. \end{aligned} \quad (5)$$

It is often convenient to choose $\epsilon_{ij}(-\mathbf{k}; \lambda) = \epsilon^*_{ij}(\mathbf{k}; \lambda)$ which implies that

$$p_{\mathbf{k},\lambda} = p^*_{-\mathbf{k},\lambda} \quad (6)$$

in Eq. (4). This brings the Einstein-Hilbert action for tensor modes to the following form:

$$\begin{aligned} S_T^{(2)} &= \sum_{\lambda=+, \times} \int d\tau d^3\mathbf{k} \\ &\times \left((\partial_\tau |p_{\mathbf{k},\lambda}|)^2 - \left(k^2 - \frac{a''}{a} \right) |p_{\mathbf{k},\lambda}|^2 \right). \end{aligned} \quad (7)$$

At this point, one can assume that tensor perturbations during inflation are either classical or quantum-mechanical.

Let us briefly discuss what happens when gravity is treated quantum-mechanically. In this case, the field $p_{\mathbf{k},\lambda}$ is now promoted to an operator, which can be expanded in terms of creation and annihilation operators:

$$\hat{p}_{\mathbf{k},\lambda} = p_k(\tau) \hat{a}_{\mathbf{k},\lambda} + p_k^*(\tau) \hat{a}^\dagger_{\mathbf{k},\lambda}. \quad (8)$$

The mode function $p_k(\tau)$ satisfies the following equation of motion:

$$p_k'' + \left(k^2 - \frac{a''}{a} \right) p_k = 0. \quad (9)$$

The Fourier-transformed field $\hat{p}(\tau, \mathbf{x})$ and its conjugate momentum $\hat{\pi}(\tau, \mathbf{x})$ satisfy the canonical commutation relations on hypersurfaces of constant τ :

$$\begin{aligned} [\hat{p}(\tau, \mathbf{x}), \hat{p}(\tau, \mathbf{x}')] &= 0, \quad [\hat{\pi}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{x}')] = 0, \\ [\hat{p}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{x}')] &= i\delta^3(\mathbf{x} - \mathbf{x}'), \end{aligned} \quad (10)$$

which is equivalent to imposing the following commutation relations on the creation and annihilation operators in Eq. (8):

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = i\delta^3(\mathbf{k} - \mathbf{k}'). \quad (11)$$

These relation enforce the following Wronskian condition on the mode function $p_k(\tau)$:

$$p_k^*(\tau) \frac{dp_k(\tau)}{d\tau} - p_k(\tau) \frac{dp_k^*(\tau)}{d\tau} = -i. \quad (12)$$

In a de Sitter background, where $a(\tau) = -1/(H\tau)$, H being the Hubble rate of expansion of the Universe, the solution to Eq. (9) is given by

$$p_k(\tau) = \alpha_k (-\tau)^{1/2} H_{3/2}^{(1)}(-k\tau) - \beta_k (-\tau)^{1/2} H_{3/2}^{(2)}(-k\tau), \quad (13)$$

where $H_{3/2}^{(1)}(x)$ and $H_{3/2}^{(2)}$ are the Hankel functions of order 3/2. In the infinite past, $k\tau \rightarrow -\infty$, the mode function $p_k(\tau)$ behaves like

$$p_k(\tau) = -\alpha_k \sqrt{\frac{2}{k\pi}} e^{-ik\tau} + \beta_k \sqrt{\frac{2}{k\pi}} e^{ik\tau}. \quad (14)$$

Using the general solution given by Eq. (13), the Wronskian condition, Eq. (12), implies that

$$|\alpha_k|^2 - |\beta_k|^2 = \frac{\pi}{4}. \quad (15)$$

In the standard theory of cosmological perturbations, it is usually assumed that the modes approached the Bunch-Davies vacuum at infinite past (see, e.g., [7])²

$$p_k(\tau) \rightarrow \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad \text{for } k\tau \rightarrow -\infty \quad (16)$$

when the wavelength of the mode is much smaller than the Hubble radius. This will result in the following values for the coefficients in Eq. (14):

$$\alpha_k = -\frac{\sqrt{\pi}}{2}, \quad \beta_k = 0. \quad (17)$$

The power spectrum of the gravitational waves for the tensor modes can be computed in the limit that the mode is well outside the Hubble patch:

$$\mathcal{P}_T = 2 \left(\frac{32\pi}{M_p^2} \right) \frac{k^3}{2\pi^2} \left| \frac{p_k(\tau)}{a(\tau)} \right|_{\frac{k}{aH} \rightarrow 0}^2, \quad (18)$$

where the factor of two counts the two helicities of the tensor mode. In the standard case with the amplitudes given by Eq. (17) and for a de Sitter background with

² This is based on the *sole* assumption that no new physics appears at very small scales (see, for example, [10]).

$k/aH = -k\tau$, we obtain from Eq. (18) the following power spectrum:

$$\mathcal{P}_T^{\text{quantum}} = \frac{16H^2}{\pi M_p^2}, \quad (19)$$

where H denotes the Hubble expansion rate of the Universe during inflation.

Let us now pause here and ask what would be different if we were to treat the gravitational waves *classically*.

First of all, we cannot expand a classical field $p_{\mathbf{k},\lambda}$ in terms of creation and annihilation operators as in Eq. (8). However, it still satisfies the equation of motion determined by Eq. (9):

$$p''_{\mathbf{k},\lambda} + \left(k^2 - \frac{a''}{a}\right) p_{\mathbf{k},\lambda} = 0, \quad (20)$$

which in a de Sitter background has a solution similar to Eq. (13):

$$p_{\mathbf{k},\lambda}(\tau) = \alpha_{\mathbf{k},\lambda}(-\tau)^{1/2} H_{3/2}^{(1)}(-k\tau) - \beta_{\mathbf{k},\lambda}(-\tau)^{1/2} H_{3/2}^{(2)}(-k\tau). \quad (21)$$

The main difference as compared to the quantum case is that we cannot impose the commutation relations, Eqs. (10,11), on classical fields. As a consequence, the Wronskian condition given by Eq. (12) is no longer valid for classical gravitational waves. In other words, the classical amplitudes $\alpha_{\mathbf{k},\lambda}$ and $\beta_{\mathbf{k},\lambda}$ in Eq. (21) do not have to obey the relationship given by Eq. (15).

Nonetheless, one has to ensure that the mode function P^i_j is real, i.e., $p_{\mathbf{k},\lambda}$ satisfies Eq. (6), which imposes the following condition on the classical amplitudes:

$$\alpha_{\mathbf{k},\lambda} = -\beta_{-\mathbf{k},\lambda}^*. \quad (22)$$

In a homogeneous and an isotropic background, the tensor perturbation of the metric cannot be sourced by the matter perturbations at the first order. Also, at a linear order there is no source term for the metric $h_{\mu\nu}$. Therefore, in a classical gravity, there is no reason *a priori* why $\alpha_{\mathbf{k},\lambda}, \beta_{\mathbf{k},\lambda} \neq 0$, unlike in the quantum case, where the Wronskian condition given by Eq. (15) prevents both the amplitudes to be zero simultaneously. Hence in a classical gravity, the amplitude of primordial gravitational waves generated during inflation can in principle be absolutely zero, i.e. $\alpha_{\mathbf{k},\lambda} = 0 = \beta_{\mathbf{k},\lambda}$, unless there are other source terms for $h_{\mu\nu}$ arising from higher order perturbations. The initial conditions for the gravitational wave solution could then be set by these higher order

corrections, but their contribution is known to be negligible [11].

The classical power spectrum for the gravitational waves can now be computed from the formula given by Eq. (18), with $p_k(\tau)$ replaced by $p_{\mathbf{k},\lambda}$ given by Eq. (21) and we obtain

$$\mathcal{P}_T^{\text{classical}} = \frac{64 |\alpha_{\mathbf{k},\lambda} + \beta_{\mathbf{k},\lambda}|^2 H^2}{M_p^2 \pi^2}, \quad (23)$$

where $\alpha_{\mathbf{k},\lambda}, \beta_{\mathbf{k},\lambda}$ can be arbitrary, as long as they satisfy the reality condition given by Eq. (22).

In what follows, we discuss some of the implications of the above results for detecting tensor modes. It is generically assumed that a positive detection of primordial gravitational waves via tensor modes would naturally put a bound on the scale of inflation. It must be emphasized that this is a correct statement *only if* gravity were treated quantum-mechanically so that the power spectrum is solely determined by the Hubble expansion rate of the Universe during inflation as in Eq. (19). Requiring that it must satisfy the current observational constraint from WMAP [2], i.e.,

$$\mathcal{P}_T^{\text{quantum}} = \frac{16H^2}{\pi M_p^2} \approx \frac{16\rho_{\text{inf}}}{M_p^4} \lesssim 10^{-10}, \quad (24)$$

we obtain an upper bound on the scale of inflation,

$$\rho_{\text{inf}}^{1/4} \sim V_{\text{inf}}^{1/4} \lesssim 10^{16.6} \text{ GeV}, \quad (25)$$

which is around the grand unified theory (GUT) scale.

On the other hand, for the case of classical gravity, as shown in Eq. (23), the gravitational power spectrum also depends on the hitherto unknown amplitudes $\alpha_{\mathbf{k},\lambda}, \beta_{\mathbf{k},\lambda}$ of the initial classical configuration. If there is no *initial* condition for the classical gravitational waves, the amplitude of the gravity-wave spectrum could be arbitrarily small, *irrespective* of the scale of inflation. This implies that the scale of inflation could be high (may be GUT-scale or even beyond) but one would not be able to detect their signature in the CMB radiation. This result has a profound effect on the inflationary model-building.

Note however that for non-zero initial classical amplitudes, the energy density of the gravitational waves could put some constraint on the coefficients $\alpha_{\mathbf{k},\lambda}, \beta_{\mathbf{k},\lambda}$. Classically the energy density of the gravitational wave profile when the modes are inside the Hubble patch, i.e.,

for $k^2 \gg a''/a$, is given by

$$\begin{aligned} \langle T_{00} \rangle &= \sum_{\lambda=+, \times} \left\langle \int d\tau d^3\mathbf{k} \left((\partial_\tau |p_{\mathbf{k},\lambda}|)^2 + k^2 |p_{\mathbf{k},\lambda}|^2 \right) \right\rangle \\ &\propto M_p^2 \sum_{\lambda=+, \times} \int_{k_i}^{k_f} k dk \left(|\alpha_{\mathbf{k},\lambda}|^2 + |\beta_{\mathbf{k},\lambda}|^2 \right), \quad (26) \end{aligned}$$

where the second line is obtained from the fact that $\int d\tau e^{-2ik\tau}$ over time is zero. This energy density should be negligible in comparison with the energy density that drives inflation, $\rho_{\text{inf}} \sim H^2 M_p^2$. Thus, in principle, the scale of inflation could be larger than the GUT scale, i.e., $\rho_{\text{inf}}^{1/4} \sim V_{\text{inf}}^{1/4} \gtrsim 10^{16}$ GeV, provided $|\alpha_{\mathbf{k},\lambda}|^2 + |\beta_{\mathbf{k},\lambda}|^2 \ll 1$ for the range of \mathbf{k} 's that are excited in the initial classical configuration.

In contrast, if primordial tensor modes are never detected, then this would mean that the initial amplitude of these stochastic gravitational waves could be very tiny irrespective of the scale of inflation.

In conclusion, a positive detection of the tensor modes in the CMB spectrum would have profound implications, which will not only put inflation on firm footing, but will also shed light on the very nature of space-time by measuring the amplitude of these tensor modes. This will be a huge step forward in resolving the long-standing issue of whether the fabric of space-time gravity should be treated as classical or quantum.

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